From Poincaré to affine invariance: How does the Dirac equation generalize?

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Abstract

A generalization of the Dirac equation to the case of affine symmetry, with $\overline{SL}(4,\mathbb{R})$ replacing $\overline{SO}(1,3)$, is considered. A detailed analysis of a Dirac-type Poincaré-covariant equation for any spin j is carried out, and the related general interlocking scheme fulfilling all physical requirements is established. Embedding of the corresponding Lorentz fields into infinite-component $\overline{SL}(4,\mathbb{R})$ fermionic fields, the constraints on the $\overline{SL}(4,\mathbb{R})$ vector-operator generalizing Dirac's γ matrices, as well as the minimal coupling to (Metric-)Affine gravity are studied. Finally, a symmetry breaking scenario for $\overline{SA}(4,\mathbb{R})$ is presented which preserves the Poincaré symmetry.

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1 Introduction

The outstanding success of the Dirac equation is unprecedented. It is a Poincaré invariant linear field equation which describes relativistic spin $\frac{1}{2}$ particles. Interactions can naturally be introduced by the minimal coupling prescription. In particular, already in the early stage of its applications, the coupling to the electro-magnetic field led to many experimental verifications. Nowadays, it represents one of the key-stones of the Standard model of Electro-Weak and Strong interactions of elementary particles. In this paper we go however beyond Poincaré invariance and study *affine* invariant generalizations of the Dirac equation, i.e., in other words, a generalization that will describe a spinorial field in a generic curved spacetime (L_4, g), characterized by arbitrary torsion and general-linear curvature. Note that the spinorial fields in the non-affine generalizations of GR (which are based on higher-dimensional orthogonal-type generalizations of the Lorentz group) are only allowed for special spacetime configurations and fail to extend to the generic case.

As General Relativity is set upon the principle of general covariance, its fundamental group is the group of diffeomorphisms $Diff(4,\mathbb{R})$. A general-relativization of the concept of spin requires (double-valued) spinorial representations of $Diff(4,\mathbb{R})$, i.e. one is interested in single-valued representations of the double-covering $\overline{Diff}(4,\mathbb{R})$. For a long time it had been wrongly believed that only single-valued representations of the Lorentz group, vectors and tensors, have a natural extension to the group $GL(n,\mathbb{R})$. However, in 1977 Y. Ne'eman has pointed out [10] that a double-covering $\overline{GL}(n,\mathbb{R})$ does exist, for proof see Ref. [12]. The latter in turn contains spinor representations. The groups $\overline{SL}(n,\mathbb{R}) \subset \overline{GL}(n,\mathbb{R})$, $n \geq 3$, are necessarily defined in infinite dimensional vector spaces. Their representations induce those of $\overline{Diff}(n,\mathbb{R})$ [12].

In contradistinction to the tensorial case where one utilizes linear representations of the group $GL(4,\mathbb{R}) \subset Diff(4,\mathbb{R})$ both in the flat tangent (Special Relativity) and in the curved spacetime (General Relativity), there are two customary constructions that provide ways to define finite spinors in a curved spacetime [19]: i) One introduces a set of anholonomic tetrads and defines an action of the (local) Lorentz group in the tangent space, or ii) One makes use of nonlinear realizations of the $\overline{Diff}(4,\mathbb{R})$ group which are linear when restricted to the Lorentz subgroup. In both cases spinorial fields essentially "live" in the tangent spacetime.

This asymmetry of treating tensors and spinors in GR is somewhat unsatisfying from a mathematical point of view. A unified description of both tensorial and spinorial fields can only be achieved by enlarging the tangential Lorentz group to the whole linear group which, together with translations, forms the affine group. The metric-affine gauge theory of gravity [5] appears to be the natural framework for this unification.

Moreover, the very existence of the $\overline{Diff}(4,\mathbb{R})$ fundamental fermionic fields opens up new roads to studies of the Gravitational interactions of the fermionic matter at the quantum level (e.g. falling of the proton into a Black Hole, when the thorough recurrences of the proton Regge trajectory can be excited gravitationally and play an essential role).

Affine-invariant extensions of the Dirac equation have been studied in [9, 1, 5, 15]. Mickelsson [9] has constructed a $\overline{GL}(4,\mathbb{R})$ covariant extension of the Dirac equation. However, its physical interpretation is rather unclear - in particular, the physically essential questions: i) the $\overline{GL}(4,\mathbb{R})$ irreducible representations content of the overall representation space and its unitarity features, and ii) the representation content of the $\overline{SO}(1,3) \supset \overline{SO}(3)$ and/or $\overline{SO}(1,3) \supset \overline{E}(2)$ subgroup-chains that define the physical particle states were not addressed at all. Cant and Ne'eman [1] found a Dirac-type equation for manifields (infinite-component fields of $\overline{SL}(4,\mathbb{R})$) which is still Poincaré invariant. They use only a subclass of representations of $\overline{SL}(4,\mathbb{R})$, the multiplicity-free ones. Since this class does not allow a $\overline{SL}(4,\mathbb{R})$ vector operator, their field equation cannot be extended to an affine Dirac-type wave equation.

We will not derive an affine Dirac-type equation explicitly. In this paper we merely focus on its reduction under the Lorentz group, i.e. on its appearance after the symmetry breaking down to $\overline{SO}(1,3)$, as well as to the relevant requirements yielding a physically feasible theory. Nonetheless, in section 2, we review some general requirements of a $\overline{SL}(4,\mathbb{R})$ vector operator which generalizes Dirac's γ-matrices. In this context we find that the mass term in an affine equation must vanish. In sections 3 to 5, we investigate Poincaré invariant Dirac-type equations for particles with arbitrary half-integral spin. We show how the method of Gel'fand et al. [3] can be generalized to derive γ-matrices for these equations. We state a theorem which yields the minimal sets of irreducible Lorentz representations needed in such equations.

In section 6, we start our construction of a Poincaré invariant Dirac-type

wave equation for manifields. This will be an equation of the form

$$(i\eta^{\alpha\beta}X_\alpha\partial_\beta - \kappa)\Psi(x) = 0\,, \tag{1}$$

where X_α are generalized Dirac matrices. Owing to the fact that each spinorial representation of $\overline{SL}(4,\mathbb{R}) \subset \overline{GL}(4,\mathbb{R})$ contains an infinite set of Lorentz representations, the Lorentz spinor Ψ will be the infinite sum of spinors $\Psi^{(j)}$. Each spinor $\Psi^{(j)}$ is chosen in such a way that it describes a physical spin j particle and/or a resonance on a certain Regge trajectory. The matrices X_α contain on its block-diagonal the γ-matrices for fermions with spin $1/2, 3/2, 5/2$ etc. The key ingredient used in this work that accounts for the physically correct particle interpretation (e.g. proton does not get spin excited by boosting) is provided by the deunitarizing automorphisms, a special feature of the (general) linear groups [12].

In section 7 and 8, we embed the representation used in (1) into (infinitely many) particularly chosen $\overline{SL}(4,\mathbb{R})$ irreducible representations and replace the spinor Ψ by the manifield $\overset{(\mathrm{SL})}{\Psi}$. This yields a still Poincaré invariant manifield equation to which an interaction force can be coupled minimally. The latter must be gravitational – or at least gravity-like, as for example the Chromogravity interaction [14], which is seen in an effective QCD approximation in the IR region and mediated by a di-gluon chromometric field $G_{\mu\nu} \sim g_{ab}A_\mu{}^a A_\nu{}^b$ $(a,b = 1,2,\ldots,8)$. This is due to the fact that the gauge group of gravity "effectively" contains a tensor operator, the shear tensor, which is able to excite the spin in $\Delta j = 2$. In comparison with [1] we make use of *non-multiplicity-free* representations of $\overline{SL}(4,\mathbb{R})$ which allow a $\overline{SL}(4,\mathbb{R})$ vector operator.

In section 9, we summarize the steps which led to our wave equation. We also present a spontaneous symmetry breaking scenario of the (special) affine group with the physical particle content corresponding to the Poincaré subgroup unitary irreducible representations. Upon breaking $\overline{SA}(4,\mathbb{R})$ down to the Poincaré group, we demonstrate how our equation is connected to a general affine Dirac-type equation.

2 $\overline{SL}(4,\mathbb{R})$ vector operator $\widetilde{X}_\alpha$

For the construction of a Dirac-type equation, which is to be invariant under (special) affine transformations, we have two possibilities to derive the matrix elements of the generalized Dirac matrices $\widetilde{X}_\alpha$.

We can consider the defining commutation relations of a $\overline{SL}(4,\mathbb{R})$ vector operator $\widetilde{X}_\alpha$,

$$[\widetilde{X}_\gamma, M_{\alpha\beta}] = ig_{\gamma\alpha}\widetilde{X}_\beta - ig_{\gamma\beta}\widetilde{X}_\alpha, \tag{2}$$

$$[\widetilde{X}_\gamma, T_{\alpha\beta}] = ig_{\gamma\alpha}\widetilde{X}_\beta + ig_{\gamma\beta}\widetilde{X}_\alpha, \tag{3}$$

with $g_{\alpha\beta}$ being structure constants of $\overline{SL}(4,\mathbb{R})$. The generators $L_{\alpha\beta}$ of the group $\overline{SL}(4,\mathbb{R})$ can be splitted into the Lorentz generators $M_{\alpha\beta} := L_{[\alpha\beta]}$ and the shear generators $T_{\alpha\beta} := L_{(\alpha\beta)}$. We obtain the matrix elements of the generalized Dirac matrices $\widetilde{X}_\alpha$ by solving these relations for $\widetilde{X}_\alpha$ in the Hilbert space of a suitable representation of $\overline{SL}(4,\mathbb{R})$.

Alternatively, we can embed $\overline{SL}(4,\mathbb{R})$ in $\overline{SL}(5,\mathbb{R})$. Let the generators of $\overline{SL}(5,\mathbb{R})$ be $L_A{}^B$, $A, B = 0, ..., 4$. Then we define the $\overline{SL}(4,\mathbb{R})$ four-vectors $\widetilde{X}_\alpha$, and $\widetilde{Y}_\alpha$ by

$$\widetilde{X}_\alpha := L_{4\alpha}, \quad \widetilde{Y}_\alpha := L_{\alpha 4}, \qquad \alpha = 0, 1, 2, 3\,. \tag{4}$$

The operator $\widetilde{X}_\alpha$ ($\widetilde{Y}_\alpha$) obtained in this way fulfills the relations (2) and (3) by construction. It is interesting to point out that the operator $\widetilde{G}_\alpha = \frac{1}{2}(\widetilde{X}_\alpha - \widetilde{Y}_\alpha)$ satisfies

$$[\widetilde{G}_\alpha, \widetilde{G}_\beta] = -iM_{\alpha\beta}\,, \tag{5}$$

thereby generalizing a property of Dirac's γ-matrices. Since $\widetilde{X}_\alpha$, $M_{\alpha\beta}$ and $T_{\alpha\beta}$ form a closed algebra, the application of $\widetilde{X}_\alpha$ on the $\overline{SL}(4,\mathbb{R})$ states does not lead out of the $\overline{SL}(4,\mathbb{R})$ representation Hilbert space.

In order to obtain an impression about the general structure of the matrix $\widetilde{X}_\alpha$, let us consider the following embedding of three finite (tensorial) representations of $SL(4,\mathbb{R})$ into one of $SL(5,\mathbb{R})$,

$$\begin{aligned} &SL(5,\mathbb{R}) \supset SL(4,\mathbb{R}) \\ &\quad \underbrace{\overset{15}{\square\square}}_{\varphi_{AB}} \supset \underbrace{\overset{10}{\square\square}}_{\varphi_{\alpha\beta}} \oplus \underbrace{\overset{4}{\boxtimes\square}}_{\varphi_\alpha} \oplus \underbrace{\overset{1}{\boxtimes\boxtimes}}_{\varphi}, \end{aligned} \tag{6}$$

where $\square$ is the Young tableau for an irreducible vector representation of $SL(n,\mathbb{R})$, $n=4,5$. The effect of the application of the $SL(4,\mathbb{R})$ vector $\widetilde{X}_\alpha$ on the fields φ, φ_α and $\varphi_{\alpha\beta}$ is

$$\begin{aligned}
&\overset{\widetilde{X}_\alpha}{\square} \otimes \overset{\varphi}{\boxtimes\boxtimes} = \overset{\varphi_\alpha}{\square} \\
&\overset{\widetilde{X}_\alpha}{\square} \otimes \overset{\varphi_\alpha}{\boxtimes\square} = \overset{\varphi_{\alpha\beta}}{\square\square} \\
&\overset{\widetilde{X}_\alpha}{\square} \otimes \overset{\varphi_{\alpha\beta}}{\square\square} = 0 \,.
\end{aligned} \tag{7}$$

Other possible Young tableaux do not appear due to the closure of the Hilbert space. Gathering these fields in a vector $\varphi_M = (\varphi, \varphi_\alpha, \varphi_{\alpha\beta})^{\mathrm{T}}$, from (7) we can read off the structure of $\widetilde{X}_\alpha$,

$$\widetilde{X}_\alpha = \left[\begin{array}{c|c|c}
0 & & \\
\hline
\begin{matrix} \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \\ \mathrm{x} \end{matrix} & \mathbf{0}_4 & \\
\hline
 & \begin{matrix} \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \\ \mathrm{xxxx} \end{matrix} & \mathbf{0}_{10}
\end{array}\right] . \tag{8}$$

It is interesting to observe that $\widetilde{X}_\alpha$ has zero matrices on the block-diagonal which implies that the mass operator κ in an affine invariant equation of the type (1) must vanish.

This can be proven for a general finite representation of $SL(4,\mathbb{R})$. Let us consider the action of a vector operator on an arbitrary irreducible representation $D(g)$ of $SL(4,\mathbb{R})$ labeled by $[\lambda_1, \lambda_2, \lambda_3]$,

$$\begin{aligned}
[\lambda_1, \lambda_2, \lambda_3] \otimes [1,0,0] = {} & [\lambda_1 + 1, \lambda_2, \lambda_3] \oplus [\lambda_1, \lambda_2 + 1, \lambda_3] \oplus \\
& [\lambda_1, \lambda_2, \lambda_3 + 1] \oplus [\lambda_1 - 1, \lambda_2 - 1, \lambda_3 - 1] \,.
\end{aligned} \tag{9}$$

None of the resulting representations agrees with the representation $D(g)$ nor with the *contragradient* representation $D^{\mathrm{T}}(g^{-1})$ given by

$$[\lambda_1, \lambda_2, \lambda_3]^{\mathrm{c}} = [\lambda_1, \lambda_1 - \lambda_3, \lambda_1 - \lambda_2] \,. \tag{10}$$

For a general (reducible) representation this implies vanishing matrices on the block-diagonal of $\widetilde{X}_\alpha$ by similar argumentation as (7) led to the structure (8). Let the representation space be spanned by $\Phi = (\varphi_1, \varphi_2, ...)^{\mathrm{T}}$ with φ_i irreducible. Now we consider the Dirac-type equation (1) in the rest frame $p_\mu = (E_{(0)}, 0, 0, 0)$ restricted to the subspace spanned by φ_i,

$$E_{(0)} \langle \varphi_i | \widetilde{X}^0 | \varphi_i \rangle = \langle \varphi_i | \kappa | \varphi_i \rangle = m_i \,, \tag{11}$$

where we assumed the operator κ to be diagonal. So the mass m_i and therewith κ must vanish since $\langle \varphi_i | \widetilde{X}^0 | \varphi_i \rangle = 0$. Therefore, in an affine invariant Dirac-type wave equation, the mass generation is dynamical, i.e. it can only be evoked by an interaction. This agrees with the fact that the Casimir operator of the special affine group $\overline{SA}(4, \mathbb{R})$ vanishes leaving the masses unconstrained [8]. So we believe that our statement also holds for infinite representations of $\overline{SL}(4, \mathbb{R})$.

3 Prerequisites from the representation theory of the Lorentz group [3]

In the following three sections we want to find a Dirac-type equation for particles with arbitrary half-integral spin. Our main concern will be the construction of the generalized Dirac matrices X_α. The wave equation should be invariant with respect to Poincaré transformations. This implies that X_α shall be a Lorentz vector operator satisfying

$$[X_\gamma, M_{\alpha\beta}] = i g_{\gamma\alpha} X_\beta - i g_{\gamma\beta} X_\alpha \,, \tag{12}$$

with $M_{\alpha\beta}$ being the Lorentz generators. We obtain the matrix elements of the generalized Dirac matrices X_α by solving these relations for X_α in the Hilbert space of a suitable representation of $\overline{SO}(1, 3)$.

Determination of X_α by the method of Gel'fand

The representations of the Lorentz subgroup $\overline{SO}(1, 3)$ can either be labeled by $\tau = [l_0, l_1]$ or by $D(j_1, j_2)$. These labels are related by

$$l_0 = j_1 - j_2, \quad l_1 = j_1 + j_2 + 1, \tag{13}$$

with j_1 and j_2 being the eigenvalues of the Casimir operators of $SU(2) \times SU(2) \simeq \overline{SO}(1,3)$. The total angular momentum l is constrained by

$$|j_1 - j_2| \leq l \leq j_1 + j_2, \quad \text{i.e. } |l_0| \leq l \leq l_1 - 1. \tag{14}$$

Two representations $\tau = [l_0, l_1]$ and $\tau' = [l'_0, l'_1]$ are *coupled* by X_α when

$$[l'_0, l'_1] = [l_0 \pm 1, l_1] \quad \text{(type A)}\,, \tag{15}$$
$$[l'_0, l'_1] = [l_0, l_1 \pm 1] \quad \text{(type B)}. \tag{16}$$

We depicted them by the *interlocking* scheme:

$$\tau \longleftrightarrow \tau'.$$

Assume some irreducible Lorentz representations are given. Gel'fand et al. [3] p.271-277 have solved (2) for X_α. They find the matrix elements of X_0 to be of the form

$$\left\langle \begin{matrix} j'_1 & j'_2 \\ l' & m' \end{matrix} \middle| X_0 \middle| \begin{matrix} j_1 & j_2 \\ l & m \end{matrix} \right\rangle = c^{\tau\tau'}_{lm;l'm'} = c^{\tau\tau'}_{l} \delta_{ll'} \delta_{mm'}\,. \tag{17}$$

For $[l'_0, l'_1] = [l_0 + 1, l_1]$ the matrices $c^{\tau\tau'}_l$ $(l = |l_0|, ..., l_1 - 1)$ are given by

$$\begin{aligned} c^{\tau\tau'}_l &= c^{\tau\tau'} \sqrt{(l + l_0 + 1)(l - l_0)}, \\ c^{\tau'\tau}_l &= c^{\tau'\tau} \sqrt{(l + l_0 + 1)(l - l_0)}, \end{aligned} \tag{18}$$

and for $[l'_0, l'_1] = [l_0, l_1 + 1]$ by

$$\begin{aligned} c^{\tau\tau'}_l &= c^{\tau\tau'} \sqrt{(l + l_1 + 1)(l - l_1)}, \\ c^{\tau'\tau}_l &= c^{\tau'\tau} \sqrt{(l + l_1 + 1)(l - l_1)}, \end{aligned} \tag{19}$$

and $c^{\tau\tau'}_l = c^{\tau'\tau}_l = 0$ for non-interlocking representations τ and τ'. $c^{\tau\tau'}$ and $c^{\tau'\tau}$ are arbitrary complex numbers. The matrix elements of X_1, X_2 and X_3 can be derived straight-forwardly from X_0, see [3], p. 276f.

Requirements on the Lorentz representations

Which class of irreducible representations are suitable for the description of fermions? Gel'fand et al. [3] impose the following requirements on the Dirac-type equation (1):

a) It shall be invariant under space reflections. An irreducible representation of the *complete* Lorentz group induces a representation of the *proper* Lorentz group. This representation is either irreducible (Case I) or it breaks up into two irreducible pieces (Case II). In the first case we have $\tau = \dot{\tau}$, where $\dot{\tau} = \pm[l_0, -l_1]$ is the *conjugate* representation of τ. In the second case, $\tau \oplus \tau'$, we have $\tau' = \dot{\tau}$ and the condition $c^{\tau\tau'} = c^{\dot{\tau}\dot{\tau}'}$ for the parameters in X_0.

b) There shall exist a non-degenerate invariant Hermitean form. This guarantees that Eq. (1) can be derived from a Lagrangian. One requires that $\tau = \tau^*$ or $\dot{\tau} = \tau^*$, where $\tau^* = [l_0, -\bar{l}_1]$ is the adjoint representation of τ. For the parameters $c^{\tau\tau'}$ we have the condition $c^{\tau\tau'} = \pm\bar{c}^{\tau'^*\tau^*}$.

The requirements a) and b) impose constraints on the labels l_0 and l_1 of the representations $\tau = [l_0, l_1]$. They are satisfied by the representations

$$[\tfrac{1}{2}, l_1] \oplus [-\tfrac{1}{2}, l_1], \quad l_1 \text{ real}. \tag{20}$$

c) The particle shall have positive probability (positive "charge"), i.e.

$$\int J_0 \, d^3x = \int \overline{\Psi} X_0 \Psi \, d^3x > 0, \tag{21}$$

and energy of both signs in order to describe particles and antiparticles. Gel'fand's method guarantees this by requiring X_0 to have eigenvalues ± 1 for states corresponding to the spin of the particle and vanishing eigenvalues for lower spin components. This will be demonstrated in the following example.

4 Determination of X_α exemplified at a spin 5/2 field

Let us determine the matrix elements of X_0 for a fermion with spin 5/2. We follow Gel'fand et al. [3] who determined this matrix for a spin 3/2 particle. A spin 5/2 particle is described by the four representations $\tau_1 = \bar{\tau}_1 = [\frac{1}{2}, \frac{3}{2}]$, $\tau_2 = [\frac{1}{2}, \frac{5}{2}]$ and $\tau_3 = [\frac{1}{2}, \frac{7}{2}]$ and their conjugate representations. τ_3 describes a composite system of particles with spin 1/2, 3/2 and 5/2. The representations τ_1, $\bar{\tau}_1$ and τ_2 are necessary to eliminate components with spin 1/2 and 3/2 which are introduced by τ_3. Fig. 1 shows the interlocking scheme[1] of these representations. We indicate the presence of two representations of the same type by a double arrow.

[1]For simplicity, arrows indicating interlockings are replaced by lines.

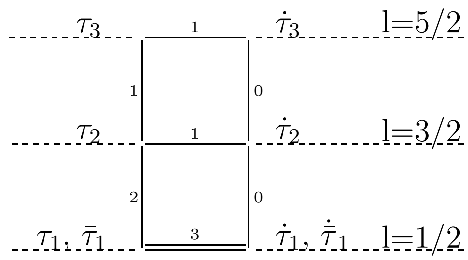


Figure 1: Interlocking scheme for a spin-5/2 particle.

We now want to determine the compartment matrices $c_l^{\tau\tau'}$ which form the Dirac-type matrix $X_0^{(j=5/2)}$, see Eq. (17). From the requirement of parity invariance we obtain:

$$
\begin{aligned}
&c^{\tau_1\dot\tau_1} = c^{\dot\tau_1\tau_1}, \quad c^{\bar\tau_1\dot{\bar\tau}_1} = c^{\dot{\bar\tau}_1\bar\tau_1}, \quad c^{\tau_2\dot\tau_2} = c^{\dot\tau_2\tau_2}, \quad c^{\tau_3\dot\tau_3} = c^{\dot\tau_3\tau_3}, \\
&c^{\tau_1\dot{\bar\tau}_1} = c^{\dot\tau_1\bar\tau_1}, \quad c^{\tau_1\tau_2} = c^{\dot\tau_1\dot\tau_2}, \quad c^{\tau_2\tau_3} = c^{\dot\tau_2\dot\tau_3}, \quad c^{\bar\tau_1\tau_2} = c^{\dot{\bar\tau}_1\dot\tau_2}, \\
&c^{\dot{\bar\tau}_1\tau_1} = c^{\bar\tau_1\dot\tau_1}, \quad c^{\tau_2\tau_1} = c^{\dot\tau_2\dot\tau_1}, \quad c^{\tau_3\tau_2} = c^{\dot\tau_3\dot\tau_2}, \quad c^{\tau_2\bar\tau_1} = c^{\dot\tau_2\dot{\bar\tau}_1}\,.
\end{aligned}
\tag{22}
$$

From the requirement of the existence of a Hermitean form we get

$$
\begin{aligned}
&c^{\tau_1\dot\tau_1} = \bar c^{\tau_1\dot\tau_1}, \quad c^{\bar\tau_1\dot{\bar\tau}_1} = \bar c^{\bar\tau_1\dot{\bar\tau}_1}, \quad c^{\tau_2\dot\tau_2} = \bar c^{\tau_2\dot\tau_2}, \quad c^{\tau_3\dot\tau_3} = \bar c^{\tau_3\dot\tau_3}, \\
&c^{\tau_1\dot{\bar\tau}_1} = \pm\bar c^{\bar\tau_1\dot\tau_1}, \quad c^{\tau_1\tau_2} = \pm\bar c^{\dot\tau_2\dot\tau_1}, \quad c^{\tau_2\tau_3} = \pm\bar c^{\dot\tau_3\dot\tau_2}, \quad c^{\bar\tau_1\tau_2} = \pm\bar c^{\dot\tau_2\dot{\bar\tau}_1}.
\end{aligned}
\tag{23}
$$

Using (17) we now compute the compartment matrices $c_l^{\tau\tau'}$ for $l = 1/2, 3/2$, $5/2$ while taking into account the above relations between the parameters $c^{\tau\tau'}$. Computer algebra yields [7]:

$$c_{5/2}^{\tau\tau'} = \begin{array}{cc} \tau_3 & \dot{\tau}_3 \\ \end{array} \left[\begin{array}{cc} 0 & 3g \\ 3g & 0 \end{array}\right], \quad c_{3/2}^{\tau\tau'} = \begin{array}{c} \begin{array}{cccc} \tau_2 & \dot{\tau}_2 & \tau_3 & \dot{\tau}_3 \end{array} \\ \left[\begin{array}{cccc} 0 & 2e & \sqrt{\frac{5}{8}}f & 0 \\ 2e & 0 & 0 & \sqrt{\frac{5}{8}}f \\ -\sqrt{\frac{5}{8}}f & 0 & 0 & 2g \\ 0 & -\sqrt{\frac{5}{8}}f & 2g & 0 \end{array}\right] \end{array},$$

$$c_{1/2}^{\tau\tau'} = \begin{array}{c} \begin{array}{cccccccc} \tau_1 & \dot{\tau}_1 & \bar{\tau}_1 & \dot{\bar{\tau}}_1 & \tau_2 & \dot{\tau}_2 & \tau_3 & \dot{\tau}_3 \end{array} \\ \left[\begin{array}{cccccccc} 0 & a & 0 & b & h & 0 & 0 & 0 \\ a & 0 & b & 0 & 0 & h & 0 & 0 \\ 0 & -b & 0 & c & d & 0 & 0 & 0 \\ -b & 0 & c & 0 & 0 & d & 0 & 0 \\ h & 0 & d & 0 & 0 & e & f & 0 \\ 0 & h & 0 & d & e & 0 & 0 & f \\ 0 & 0 & 0 & 0 & -f & 0 & 0 & g \\ 0 & 0 & 0 & 0 & 0 & -f & g & 0 \end{array}\right] \end{array},$$

where

$$a = \frac{-1}{3}; b = \frac{1}{24}\sqrt{10}; c = \frac{1}{3}; d = \frac{9}{40}\sqrt{10}; e = -\frac{1}{3}; f = \frac{4}{15}\sqrt{10}; g = \frac{1}{3}; h := 0.$$

We excluded particles with spin 1/2 and 3/2 by requiring additionally that all eigenvalues of the matrices $c_{1/2}^{\tau\tau'}$ and $c_{3/2}^{\tau\tau'}$ are zero. The eigenvalues of $c_{5/2}^{\tau\tau'}$ must be ± 1 in order to have both particles and antiparticles with spin 5/2.

The compartment matrix $c_{1/2}^{\tau\tau'}$ contains eight parameters. Three of them, namely e, f and g, are already fixed by the matrices $c_{3/2}^{\tau\tau'}$ and $c_{5/2}^{\tau\tau'}$. We can set one parameter equal to zero (here $h = 0$) since the requirement of vanishing eigenvalues fixes only four parameters in the matrix $c_{1/2}^{\tau\tau'}$. If we had taken just one representation of the type $\tau_1 = [\frac{1}{2}, \frac{3}{2}]$, we would have had only $5 - 3 = 2$ free parameters in $c_{1/2}^{\tau\tau'}$. However, in this case 3 parameters will be fixed by the requirement of vanishing eigenvalues.

5 Lorentz representation of a fermionic particle

The method of Gel'fand et al. can be generalized for all fermions with spin j. For that we have to use an irreducible Lorentz representation which contains j as highest spin value. We refer to this as the *main representation*. Thereby we also introduce other spin components which must be eliminated by a set of auxiliary representations. The following theorem helps us to find these representations:

Theorem 1 *The general interlocking scheme for a particle with arbitrary half-integral spin j reads*

$$\begin{array}{ccccccccccccc}
\tau_1 & \cdots & \tau_{n-4} & \text{———} & \tau_{n-3} & \text{———} & \tau_{n-2} & \text{———} & \tau_{n-1} & \text{———} & \tau_n & \text{———} & \tau_{n+1} \\
\vdots & & ||| & & || & & || & & || & & | & & | \\
\dot{\tau}_1 & \cdots & \dot{\tau}_{n-4} & \text{———} & \dot{\tau}_{n-3} & \text{———} & \dot{\tau}_{n-2} & \text{———} & \dot{\tau}_{n-1} & \text{———} & \dot{\tau}_n & \text{———} & \dot{\tau}_{n+1}
\end{array}$$

where $\tau_i = [\frac{1}{2}, i + \frac{1}{2}]$ and $\dot{\tau}_i = [-\frac{1}{2}, i + \frac{1}{2}]$ $(i = 1, ..., n+1; n = j - \frac{1}{2})$ are finite irreducible representations of the Lorentz group. Let us denote the corresponding representation by ρ_j. The number M_i of vertical arrows between τ_i and $\dot{\tau}_i$ is the multiplicity with which they occur in ρ_j.

Remarks:

i) The representations ρ_j satisfy the requirements a) - c) of Section 3, i.e. Eq. (1) is parity invariant, derivable from a Lagrangian and describes both particles and antiparticles with spin j.

ii) The spin content of the main representation $\tau_{n+1} = [\frac{1}{2}, j+1]$ is $\frac{1}{2}, \frac{3}{2}, ..., j$, see Eq. (14). The other representations $\tau_1, ..., \tau_n$ are needed to eliminate lower spin values such that only a particle with spin j remains.

iii) In Eq. (1) we take the field $\Psi^{(j)} := (\psi^{(1)}, ..., \psi^{(i)}, ..., \psi^{(n+1)})^T$, where $\psi^{(i)}$ $(i = 1, ..., n+1)$ denotes a spinor with (sum over all j values, see ii))

$$\sum_{j=1/2}^{i-1/2} 2 \underbrace{(2j+1)}_{\text{m}-\text{degeneracy}} = \sum_{l=1}^{i} 4j = 2i(i+1) \tag{24}$$

components. We note that some spinors $\psi^{(i)}$ occur several times in $\Psi^{(j)}$ according to their multiplicities M_i.

iv) The above interlocking scheme corresponds to the representation

$$
\begin{array}{rcc}
\rho_j := & D(\frac{1}{2}(n+1),\frac{1}{2}n) \oplus D(\frac{1}{2}n,\frac{1}{2}(n+1)) & \\
\oplus & D(\frac{1}{2}n,\frac{1}{2}(n-1)) \oplus D(\frac{1}{2}(n-1),\frac{1}{2}n) & \\
\oplus & 2[D(\frac{1}{2}(n-1),\frac{1}{2}(n-2)) \oplus D(\frac{1}{2}(n-2),\frac{1}{2}(n-1))] & \\
\oplus & 2[D(\frac{1}{2}(n-2),\frac{1}{2}(n-3)) \oplus D(\frac{1}{2}(n-3),\frac{1}{2}(n-2))] & (25) \\
\oplus & 2[D(\frac{1}{2}(n-3),\frac{1}{2}(n-4)) \oplus D(\frac{1}{2}(n-4),\frac{1}{2}(n-3))] & \\
\oplus & 3[D(\frac{1}{2}(n-4),\frac{1}{2}(n-5)) \oplus D(\frac{1}{2}(n-5),\frac{1}{2}(n-4))] & \\
 & \vdots & \\
\oplus & M_1[D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})]. &
\end{array}
$$

We will now prove the theorem. The representation (25) shows that the multiplicities M_i do not follow a simple rule. The proof provides an algorithm for the determination of these multiplicities.

Proof of the theorem

First, let us assume that a diagram[2] of the type of Theorem 1 is given and we want to verify whether it has the right number of multiplicities or not. We have to assure that there are enough parameters to fix in each compartment matrix since then we are able to set the parameters such that the eigenvalues of the compartment matrices are all zero and ± 1 for the compartment matrix with the highest l-value, respectively.

This can be achieved in the following way. We write the highest l-value $(= l_1 - 1)$ of each representation τ_i next to it, see e.g. Fig. 1 and Fig. 2. Note that the partial diagram above an l-value contains all the information of the number of parameters of the compartment matrix $c_l^{\tau\tau'}$ with this l-value. It determines the number of free parameters A_l and the number of parameters B_l which will be fixed by the method of Gel'fand.

Determination of A_l:

Observe that each interlocking gives rise to one parameter $c^{\tau\tau'}$. Actually, each interlocking gives rise to two parameters, $c^{\tau\tau'}$ and $c^{\tau'\tau}$, cf. Eqs. (18) and (19). However, for the class of representations given by Eq.(20), we have [3]

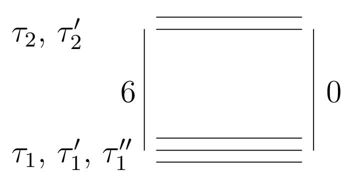
[2]For the following we rotate it in 90° anti-clockwise.

p.320 $\tau = \tau^*$, $\dot{\tau} = \dot{\tau}^*$ and thus $c^{\tau\tau'} = \pm\bar{c}^{\tau'^*\tau^*} = \bar{c}^{\tau'\tau}$ since we have to take into account the requirement that our Dirac-type equation shall be derivable from a Lagrangian, cf. [3] p.292. In other words, $c^{\tau\tau'}$ and $c^{\tau'\tau}$ are related. Thus by counting the interlockings of a partial diagram we obtain the number A_l of parameters in the compartment matrix $c_l^{\tau\tau'}$.

The number of interlockings A_l can be obtained by counting the arrows in a diagram. Horizontal arrows are weighted differently than vertical ones. The following rules can be used to determine these weights.

Rule 1 (vertical arrows): *Each vertical arrow is weighted by $n \cdot m$, whereby n and m are the multiplicities of the horizontal arrows which adjoin it. Vertical arrows between dotted representations are weighted by zero.*

Example: The following diagram shows a 2-fold and a 3-fold horizontal arrow. We weight the vertical arrow by 6 since we get the parameters $c^{\tau_1\tau_2}, c^{\tau_1\tau_2'}, c^{\tau_1'\tau_2}$, $c^{\tau_1'\tau_2'}, c^{\tau_1''\tau_2}$ and $c^{\tau_1''\tau_2'}$, i.e. 6 interlockings of type B.

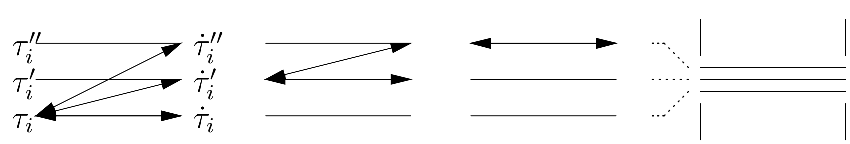


Note that, because of parity invariance, we have $c^{\tau\tau'} = c^{\dot{\tau}\dot{\tau}'}$ and therefore we must not count arrows between dotted representations. This is why we weight them by zero.

Rule 2 (horizontal arrows): *The number of interlockings (of type A) given by a n-fold arrow is*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{26}$$

Example: Let us consider a three-fold arrow (represented by three lines). We simply count the mutual interlockings of the representations τ_i and $\dot{\tau}_i$.



Here we have altogether six parameters: three parameters $c^{\tau_i\dot{\tau}_i}, c^{\tau_i\dot{\tau}_i'}, c^{\tau_i\dot{\tau}_i''}$, two parameters $c^{\tau_i'\dot{\tau}_i'}, c^{\tau_i'\dot{\tau}_i''}$ and one parameter $c^{\tau_i''\dot{\tau}_i''}$ indicated by arrows with arrowheads in the above figure. Due to parity invariance ($c^{\tau\tau'} = c^{\dot{\tau}\dot{\tau}'}$) there

are no further free parameters. A forth representation τ_i''' would interlock with each of the four conjugate representations $\dot{\tau}_i, \dot{\tau}_i', \dot{\tau}_i''$, and $\dot{\tau}_i'''$ and we would obtain four further interlockings. Clearly, a $n+1$-fold arrow has $n+1$ more interlockings than the n-fold one.

Finally, we apply

Rule 3: *A_l is the sum of the weights of all arrows in a partial diagram minus the numbers $B_{l'}$ with $l < l' \leq j$.*

When we sum up all weights we get the number of interlockings and therewith the number of free parameters in the compartment matrix $c_l^{\tau\tau'}$. We have to subtract $B_{l'}$ ($\forall\, l' > l$) since these are the number of parameters which have already been fixed by the compartment matrices $c_{l'}^{\tau\tau'}$ and are not at our disposal any more.

Determination of B_l:

Rule 4: *B_l is the number of horizontal arrows in a partial diagram — n-fold arrows are counted n times.*

This gives us the number of irreducible representations contributing to the compartment matrix $c_l^{\tau\tau'}$ and therewith its dimension $n = 2B_l$. The corresponding characteristic polynomial in λ is then of order n and has the from

$$P(\lambda) = \lambda^n + c_{n-2}\lambda^{n-2} + \cdots + c_2\lambda^2 + c_0. \tag{27}$$

Since the eigenvalues of X_0 (and therewith those of $c_l^{\tau\tau'}$) are $\pm\lambda_1, \pm\lambda_2, ...$ [2] p.144, the characteristic polynomial only contains even powers of λ. The constants c_i depend on the parameters of $c_l^{\tau\tau'}$. In order to get vanishing eigenvalues, we set the $n/2$ constants $c_i = 0$ ($i = 0, 2, ..., n-2$). These relations fix $n/2 = B_l$ parameters.

If for some l the number A_l of parameters which are at our disposal is less than the number of parameters B_l which will be fixed then the interlocking scheme fails. In this case we apply

Rule 5: *Assume the multiplicity $M_{l+\frac{1}{2}}$ of the representation $\tau_{l+\frac{1}{2}}$ is n. If $A_l < B_l$, increase the multiplicity $M_{l+\frac{1}{2}}$ in 1 by replacing the n-fold by a $n+1$-fold arrow in the diagram and check A_l and B_l again.*

We can always introduce so many representations $\tau_{l+\frac{1}{2}}$ until $A_l \geq B_l$.

Each new representation $\tau_{l+\frac{1}{2}}$ increases B_l in one. However, A_l increases in

$$\frac{(n+1)((n+1)+1)}{2} + (n+1)\,m - \frac{n(n+1)}{2} - n\,m = n+1+m > 1, \tag{28}$$

where n is the multiplicity of $\tau_{l+\frac{1}{2}}$ and m that of $\tau_{l+\frac{1}{2}+1}$, i.e. we can always achieve that $A_l \geq B_l$.

Algorithm for obtaining the multiplicities

Now we are prepared to construct a diagram which has the right multiplicities. We start from an "empty" diagram, i.e. a diagram with simple arrows everywhere. This corresponds to $j - \frac{1}{2}$ squares. Using rules 1 and 2, we write the weights next to each arrow and determine A_l and B_l according to rules 3 and 4. We begin at the top horizontal arrow of the diagram: $A_{l=j} = 1$ and $B_{l=j} = 1$ (o.k. since $A_l \geq B_l$). Next we evaluate the partial diagram for $l = j-1$: $A_{l=j-1} = 3 - 1 = 2$ and $B_{l=j-1} = 2$ (o.k.). Then $A_{l=j-2} = 5-2-1 = 2$ and $B_{l=j-2} = 3$ (not o.k.). Therefore, we apply rule 5, i.e. we set $M_{l+\frac{1}{2}=j-\frac{3}{2}} = 1 \to 2$, and get $A_{l=j-2} = 8-2-1 = 5$ and $B_{l=j-2} = 4$ (o.k.). In this way we go ahead until we reach the bottom arrow.

As in the spin-$\frac{3}{2}$-case [3] p.347f, the energy has both signs and the charge is positive definite since the compartment matrix with the highest l-value can always be chosen to have eigenvalues ± 1.

Result: By the introduction of enough auxiliary fields it is always possible to construct a wave equation (1) which fulfills the wished properties. □

In this proof we assumed in Rule 4 that all coefficients c_i of the characteristic polynomial of a compartment matrix do not vanish ($c_i \neq 0$) and that they are all different ($c_i \neq c_j$). Of course, it might be that some c_i are zero in advance or that two or more c_i are equal. Then the relations fix less than $n/2$ parameters. However, the examples show that this is usually not the case. But this is difficult to prove. So, strictly speaking, we can only prove that a scheme works, but not that another one fails. To prove the latter we have to compute also all characteristic polynomials and check whether there are c_i which coincide or vanish.

As a final remark we mention that there exists a *non-minimal* solution for the multiplicities. In the appendix we prove that a representation ρ_j with multiplicities $M_i = n+2-i$ for the representations τ_i ($i = 1, ..., n+1$) can be used for the description of a particle with spin j. So the multiplicities in our

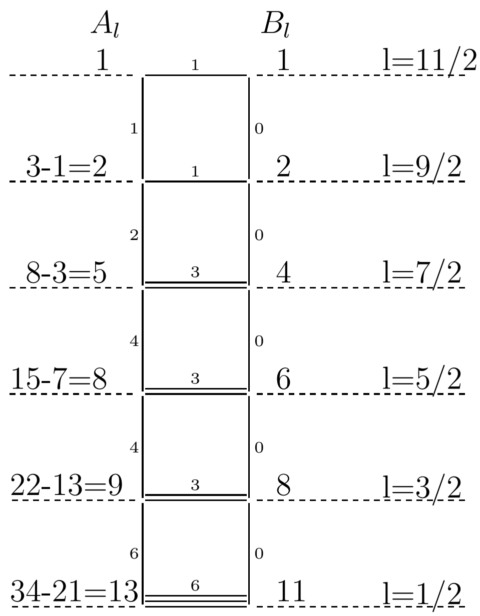


Figure 2: Diagram for a spin-11/2 particle.

minimal solution given by the above algorithm increase slower than linearly.

Comparison with the approach of Singh and Hagen [20]

The aim of our approach is the same as that of Singh-Hagen, though achieved by a completely different method: We have found a Dirac-type equation, derivable from a Lagrangian, which replaces the Rarita-Schwinger scheme of a fermion, see [20] Eq. (2). Singh-Hagen found a set of first-order differential equations which does the same. Up to spin 9/2 particles both approaches use the same Lorentz representations, cf. the representation (25) with that in [20].

The interlocking scheme for a spin-11/2-particle is shown in Fig. 2. The number next to each arrow is the number of interlockings which are induced by it, use rules 1 and 2. The above described method yields three times the representation τ_1 in contradiction to what Singh-Hagen [20] claim. If we took

τ_1 only twice, as they do, we would obtain $A_{1/2} = 29 - 21 = 8$ and $B_{1/2} = 10$ ($A < B$, not o.k.). Therefore, we have to introduce a third τ_1 representation and obtain $A_{1/2} = 34 - 21 = 13 \geq B_{1/2} = 11$ (o.k.).

6 "Reggeization"

We want to find the Lorentz representation of the resonances on hadronic Regge trajectories. These resonances can be classified by the group $\overline{SL}(4, \mathbb{R})$ [11]. When plotted in a Chew-Frautschi diagram, the Regge trajectories show a linear relation between the square of the mass M of a resonance and its spin J,

$$J = \alpha(0) + \alpha' M^2, \tag{29}$$

where $\alpha(0)$ sets the low-energy scale, about $1\,GeV$, and α' is the slope of the trajectories, about $0.9\,(GeV)^{-2}$ (numerical values for the first three flavors).

The extra-ordinary linearity of these trajectories suggests that the higher spin resonances should rather be described as excitations of the lowest state of a multiplet than by independent wave equations. For such a description we define the "Regge" representation as the direct sum of the representations ρ_j given by Theorem 1,

$$\rho := \sum_{j=\frac{1}{2}}^{\infty}{}^{\oplus} \rho_j \,. \tag{30}$$

The corresponding infinite-component spinor is $\Psi := (\Psi^{(1/2)}, \Psi^{(3/2)}, ...)^T$.

The representation ρ describes two exchange-degenerate Regge trajectories at once: the lowest state of the first one has spin $\frac{1}{2}$, the other one spin $\frac{3}{2}$. They obey the $\Delta J = 2$ rule, e.g. for spinors $\{J\} = \{\frac{1}{2}, \frac{5}{2}, ...\}$ and $\{J\} = \{\frac{3}{2}, \frac{7}{2}, ...\}$. We could also consider just one Regge trajectory. There is no crucial difference since the same irreducible Lorentz representations are used.

The irreducible representations in (25) are depicted in Fig. 3. All of them lie within the *zone of non-trivial action* of X_α. For example, the representation $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$ is indicated by a filled and an open circle at $(j_1, j_2) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

We can now apply the method of Gel'fand in order to determine the matrices $X_0^{(j)}$ for each particle with spin j. $X_0^{(1/2)}$ is equal to γ_0 used in the

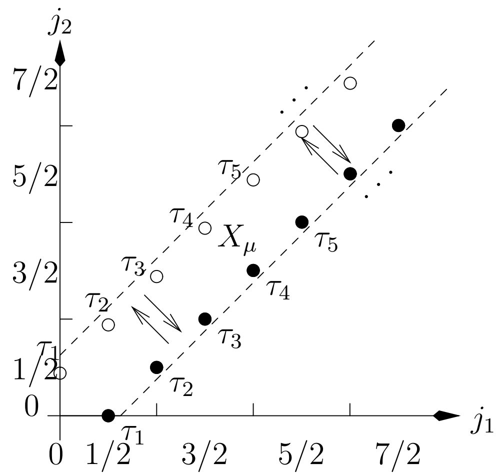


Figure 3: $(j_1,\, j_2)$-content of the Regge representation.

conventional Dirac equation. The X_0 matrix (and therewith X_a $(a = 1, 2, 3)$) corresponding to the Regge representation ρ is of the blockdiagonal form

$$X_0 = \begin{bmatrix} X_0^{(1/2)} & & & \\ & X_0^{(3/2)} & & \\ & & X_0^{(5/2)} & \\ & & & \ddots \end{bmatrix}. \tag{31}$$

Thus (1) becomes an infinite set of decoupled equations describing free Regge resonances.

We now couple the representations ρ_j in order to introduce spin excitations of the resonances. Physically such excitations of the spin can only be induced by an interaction force since the spin value does not change as long as the particle only undergoes Lorentz transformations. Two neighboring resonances on the Regge trajectories differ in their spin value in 2. We need an operator which interlocks the representations $\tau = [\frac{1}{2}, l_1]$ and $\tau' = [\frac{1}{2}, l_1 + 2]$. It turns out that this can be done by the shear operators of the group $\overline{SL}(4, \mathbb{R})$. Since the latter can only act on $\overline{SL}(4, \mathbb{R})$ manifields $\overset{\text{(SL)}}{\Psi}$, we have to embed the Regge representation into a representation of this group.

7 (Non-)multiplicity-free representations of $\overline{SL}(4, \mathbb{R})$

7.1 Multiplicity-free representations

Can we embed the Regge representation into a multiplicity-free ($k_i = 0$, $i = 1, 2$) representation of $\overline{SL}(4, \mathbb{R})$? The multiplicity-free representations are well known. They have been classified in [17]. We will not repeat this here, but we strongly recommend to study them before going ahead.

According to Harish-Chandra[3] [4] the representations $U(g)$, $g \in G$ of a noncompact group G can be defined in a homogeneous Hilbert space $H = \{f(k)|k \in K\}$ over the maximal compact subgroup $K \subset G$. Then $U(g)$ is a continuous mapping from G into the set of linear transformations on H given by

$$U(g)f(k) = \exp[\alpha(h(k, g))]f(k \cdot g), \tag{32}$$

where $g \in G$, $k \in K$, $e^h \in A$ and A is the Abelian subgroup. The maximal compact subgroup of $\overline{SL}(4, \mathbb{R})$ is $\overline{SO}(4) \simeq SU(2) \times SU(2)$. After the application of the *deunitarizing automorphism* $\mathcal{A}$ [17], the eigenvalues of its Casimir operators, j_1 and j_2, can be identified with those of the Lorentz group since $\overline{SO}(4)_{\mathcal{A}} \simeq \overline{SO}(1, 3)$. Each representation of $\overline{SL}(4, \mathbb{R})$ contains Lorentz submultiplets $(j_1,\ j_2)$. All these submultiplets are called the $(j_1,\ j_2)$-content of a $\overline{SL}(4, \mathbb{R})$ representation.

The Lorentz $(j_1,\ j_2)$ submultiplets can be excited by means of the shear operator $Z_{\alpha\beta}$ $(\alpha, \beta = 0, \pm 1)$, which is in its turn a $(1, 1)$ irreducible tensor operator of the Lorentz group. From its matrix representation in the general case, see (34) below, we deduce that its action can be visualized by a 'Union Jack', for details see [5] Ch. 4.5. In Fig. 5 this is demonstrated for the point $(7/2, 1)$. Due to the properties of the 3-j-symbols in the multiplicity-free case, we just have '$\times$'-like transitions between Lorentz submultiplets such that the lattice is divided into eight sublattices [17] Fig. 1. Four of them, L_5, L_6, L_7 and L_8, could be relevant for the embedding of the Regge representation. They are drawn in Fig. 4. However, not all of their Lorentz submultiplets belong to an *invariant* lattice, i.e. to a multiplicity-free representation of $\overline{SL}(4, \mathbb{R})$. We crossed them out in Fig. 4. By comparison with Fig. 3 we see

[3] For a summary of the representation theory of noncompact groups developed by Harish-Chandra see also [16] Sec. 3.

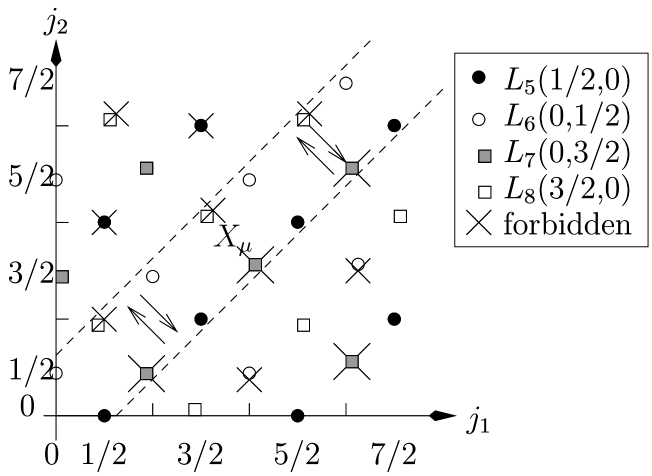


Figure 4: $(j_1,\ j_2)$-content of some multiplicity-free representations of $\overline{SL}(4,\mathbb{R})$.

that the irreducible Lorentz representations $D(n, n-1/2) \oplus D(n-1/2, n)$ $(n = 1, 2, ...)$ of the Regge representation cannot be embedded into any of the multiplicity-free representations of $\overline{SL}(4,\mathbb{R})$. Only those with $n = \frac{1}{2}, \frac{3}{2}, ...$ are contained in the lattices L_5 and L_6 and could be embedded into the $\overline{SL}(4,\mathbb{R})$ representation $D^{\text{disc}}(\frac{1}{2}, 0)_{\mathcal{A}} \oplus D^{\text{disc}}(0, \frac{1}{2})_{\mathcal{A}}$.

Moreover, we face another problem with *multiplicity-free* representations of $\overline{SL}(4,\mathbb{R})$. It is shown in [1] App. A that no multiplicity-free representation (except for the sum of ladder representations which are of no use here) admits an $\overline{SL}(4,\mathbb{R})$ vector, i.e. $(\frac{1}{2}, \frac{1}{2})$, operator $\widetilde{X}_\alpha$.

Indeed, for finite (tensorial) representations this can easily be seen by using Young tableaux. The tensor product of $\widetilde{X}_\alpha$, represented by $\Box$, and a multiplicity-free (all are ladder type) representation results in the sum of a multiplicity-free and a non-multiplicity-free representation,

$$\underbrace{\Box\Box\cdots\Box\Box}_{\text{multiplicity}-\text{free}} \otimes \Box = \underbrace{\Box\Box\cdots\Box\Box\Box}_{\text{multiplicity}-\text{free}} \oplus \underbrace{\begin{array}{l}\Box\Box\cdots\Box\Box\\ \Box\end{array}}_{\text{non}-\text{multiplicity}-\text{free}} \tag{33}$$

Consequently, the application of a $\overline{SL}(4,\mathbb{R})$ vector operator $\widetilde{X}_\alpha$ naturally leads to non-multiplicity-free representations. In the case of spinorial (infinite-dimensional) representations, we point out two relevant facts: (i) these representations are not of the ladder type, and (ii) the tensor product of the

vector representation ($\widetilde{X}_\alpha$) and a multiplicity-free spinorial irreducible representation *does not* contain any representation of the latter type. Thus, it is not possible to restrict on multiplicity-free representations alone.

7.2 Non-multiplicity-free representations

Some results for the general case can be found in [18, 19]. Here the representations are non-multiplicity-free, i.e. the label $k_i \neq 0$ $(i = 1, 2)$. The generators of $\overline{SL}(4, \mathbb{R})$, the Lorentz and shear generators, $M_{\alpha\beta}$ and $T_{\alpha\beta}$, can be replaced by the spherical tensors $J^{(1)}_\alpha$, $J^{(2)}_\alpha$, and $Z_{\alpha\beta}$ $(\alpha, \beta = 0, \pm 1)$ [17]. The matrix elements of the $SU(2)$ generators $J^{(1)}_\alpha$ and $J^{(2)}_\alpha$ are well known from angular momentum theory. The matrix elements of the shear generators $Z_{\alpha\beta}$ $(\alpha, \beta = 0, \pm 1)$ read [18]

$$\left\langle \begin{matrix} j'_1 & j'_2 \\ k'_1 m'_1 & k'_2 m'_2 \end{matrix} \right| Z_{\alpha\beta} \left| \begin{matrix} j_1 & j_2 \\ k_1 m_1 & k_2 m_2 \end{matrix} \right\rangle = (-1)^{j'_1 - m'_1} \begin{pmatrix} j'_1 & 1 & j_1 \\ -m'_1 & \alpha & m_1 \end{pmatrix} \times \\ \times (-1)^{j'_2 - m'_2} \begin{pmatrix} j'_2 & 1 & j_2 \\ -m'_2 & \beta & m_2 \end{pmatrix} \left\langle \begin{matrix} j'_1 & j'_2 \\ k'_1 & k'_2 \end{matrix} \right\| Z \left\| \begin{matrix} j_1 & j_2 \\ k_1 & k_2 \end{matrix} \right\rangle \tag{34}$$

with the reduced matrix element

$$\left\langle \begin{matrix} j'_1 & j'_2 \\ k'_1 & k'_2 \end{matrix} \right\| Z \left\| \begin{matrix} j_1 & j_2 \\ k_1 & k_2 \end{matrix} \right\rangle = \\ (-)^{j'_1 - k'_1} (-)^{j'_2 - k'_2} \tfrac{i}{2} \sqrt{(2j'_1+1)(2j'_2+1))(2j_1+1)(2j_2+1)} \times \\ \times \Bigg\{ [e + 4 - j'_1(j'_1+1) + j_1(j_1+1) - j'_2(j'_2+1) + j_2(j_2+1)] \\ \times \begin{pmatrix} j'_1 & 1 & j_1 \\ -k'_1 & 0 & k_1 \end{pmatrix} \begin{pmatrix} j'_2 & 1 & j_2 \\ -k'_2 & 0 & k_2 \end{pmatrix} \\ - (c + k_1 - k2) \begin{pmatrix} j'_1 & 1 & j_1 \\ -k'_1 & 1 & k_1 \end{pmatrix} \begin{pmatrix} j'_2 & 1 & j_2 \\ -k'_2 & -1 & k_2 \end{pmatrix} \\ - (c - k_1 + k2) \begin{pmatrix} j'_1 & 1 & j_1 \\ -k'_1 & -1 & k_1 \end{pmatrix} \begin{pmatrix} j'_2 & 1 & j_2 \\ -k'_2 & 1 & k_2 \end{pmatrix} \\ + (d + k_1 + k2) \begin{pmatrix} j'_1 & 1 & j_1 \\ -k'_1 & 1 & k_1 \end{pmatrix} \begin{pmatrix} j'_2 & 1 & j_2 \\ -k'_2 & 1 & k_2 \end{pmatrix} \\ + (d - k_1 - k2) \begin{pmatrix} j'_1 & 1 & j_1 \\ -k'_1 & -1 & k_1 \end{pmatrix} \begin{pmatrix} j'_2 & 1 & j_2 \\ -k'_2 & -1 & k_2 \end{pmatrix} \Bigg\} . \tag{35}$$

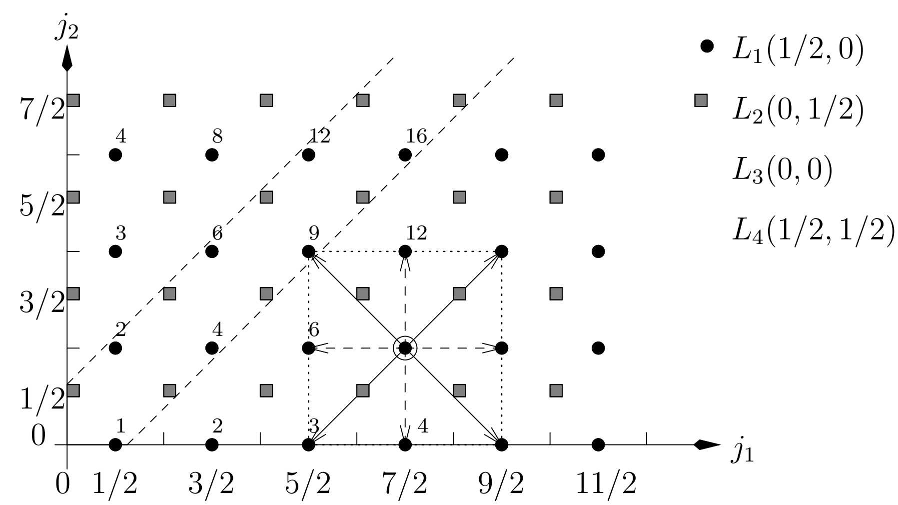


Figure 5: Four j_1-j_2-lattices - L_3 and L_4 are not needed.

In the Appendix we relate the 15 generators $L_{\alpha\beta} = M_{\alpha\beta}+T_{\alpha\beta}$ to the spherical tensors $J^{(1)}_\alpha$, $J^{(2)}_\alpha$ and $Z_{\alpha\beta}$ ($\alpha, \beta = 0, \pm 1$).

Note some differences to the multiplicity-free case. Since the operator $Z_{\alpha\beta}$ induces '$\times$'-like *and* '+'-like transitions between Lorentz submultiplets ('Union Jack'), we just have four sublattices. Two of them, $L_1(\frac{1}{2}, 0)$ and $L_2(0, \frac{1}{2})$, which are important for the embedding, are depicted in Fig. 5. Since a state is characterized by $|j_1 j_2 k_1 k_2\rangle$ and not just by $|j_1 j_2\rangle$ (quantum numbers m_1 and m_2 are ignored), we should keep in mind that we actually deal with a four-dimensional lattice. Therefore, each dot in Fig. 5 can represent more than one Lorentz submultiplet. The small-printed number next to each dot is the multiplicity of the Lorentz subrepresentation $D(j_1, j_2)$.

Determination of the multiplicities

We want to find the multiplicities of the Lorentz submultiplets of $\overline{SL}(4, \mathbb{R})$ representations. As an example, let us determine those of the lattice $L_1(\frac{1}{2}, 0)$. From the properties of the 3-j-symbols in the matrix representation of $Z_{\alpha\beta}$ we know that $k_1' - k_1 = \pm 1$ and $k_2' - k_2 = \pm 1$. This allows '$\times$'-like transitions in the k_1-k_2-lattice. It can thus be divided into eight sublattices in an analogous way as the j_1-j_2-lattice was divided in the multiplicity-free case.

We now choose two k_1-k_2-lattices such that they would form the lattice $L_1(\frac{1}{2}, 0)$, if the lattices were a j_1-j_2-lattice instead of k_1-k_2-ones. Thus the

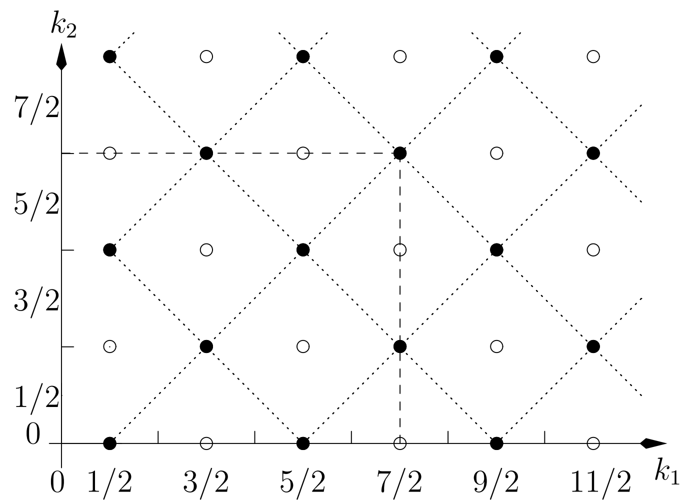


Figure 6: Two of eight k_1-k_2-lattices.

two relevant k_1-k_2-lattices are those shown in Fig. 6: one is represented by open circles, the other one by closed circles.

Now, we can ask which (j_1, j_2) submultiplets of $L_1(\frac{1}{2}, 0)$ contain a specific pair (k_1, k_2). In other words, we want to determine the number of states

$$\begin{vmatrix} j_1 & j_2 \\ k_1 & k_2 \end{matrix}\Bigg\rangle$$

for a given pair (k_1, k_2). Hereto we have to remember the conditions $j_1 \geq |k_1|$ and $j_2 \geq |k_2|$. This means that (k_1, k_2) determines the minimal value of a sublattice in the j_1-j_2-lattice in which all (j_1, j_2) submultiplets contain (k_1, k_2). In Fig. 7 we show two examples: the (j_1, j_2)-sublattices for $(k_1, k_2) = (1/2, 0)$ and $(3/2, 1)$.

In order to determine the number of a certain Lorentz submultiplet, i.e. the multiplicity of (j_1, j_2), in principle, we have to determine the sublattices of the type as in Fig. 7 for all pairs (k_1, k_2) of the k_1-k_2-lattices shown in Fig. 6. Then we count the number of sublattices which contain this (j_1, j_2) value. For short, we can also consider just $(k_1, k_2) = (j_1, j_2)$ in the k_1-k_2-lattice and count all the circles which lie inside the rectangle with the edges $(k_1, k_2) = \{(0, 0), (j_1, 0), (0, j_2), (j_1, j_2)\}$ since all of them lead to (j_1, j_2)-sublattices which contain this specific (j_1, j_2) value. In Fig. 6 this is shown for $(j_1, j_2) = (7/2, 3)$. Its multiplicity is thus 16. This is the small-printed

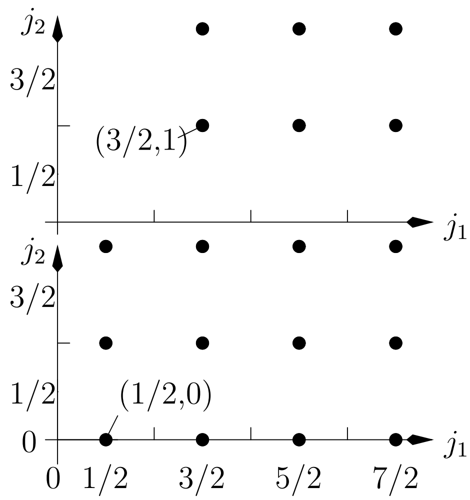


Figure 7: (j_1, j_2)-sublattices.

number next to the component $(7/2, 3)$ in Fig. 5.

We end up with a simple formula for the multiplicity m of a Lorentz submultiplet (j_1, j_2),

$$m = (j_1 + a) \times (j_2 + b), \tag{36}$$

where $a = b = \frac{1}{2}$ for half-integral and $a = b = 1$ for integral j_1, j_2 values.

8 Embedding of a Regge representation in a $\overline{SL}(4, \mathbb{R})$ representation

The $(j_1,\ j_2)$-content of the Regge representation is shown in Fig. 3. For its embedding we need a series of $\overline{SL}(4, \mathbb{R})$ which contains the j_1-j_2-lattices $L_1(\frac{1}{2}, 0)$ and $L_2(0, \frac{1}{2})$, see Fig. 5. The possible values of the complex repre-

sentation labels c, d, e in (34) are [18, 19]

$$
\begin{aligned}
&A)\, e_1 = 0,\ e_2 \in \mathbb{R}, \\
&B_1)\, d_1 = 0,\ d_2 \in \mathbb{R}, \\
&B_2)\, d_1 = \underline{k}_1 + \underline{k}_2,\ d_2 = 0; \quad \underline{k}_1 + \underline{k}_2 = \tfrac{1}{2}, 1, \tfrac{3}{2}, ..., \\
&B_3)\, 0 < d_1 < 1,\ d_2 = 0; \quad k_1 + k_2 = 0, \pm 2, \pm 4, ..., \\
&B_4)\, 0 < d_1 < \tfrac{1}{2},\ d_2 = 0; \quad k_1 + k_2 \equiv \tfrac{1}{2} (\operatorname{mod} 2) \ \text{ or } \ \tfrac{3}{2} (\operatorname{mod} 2), \\
&C_1)\, c_1 = 0,\ c_2 \in \mathbb{R}, \\
&C_2)\, c_1 = \underline{k}_1 - \underline{k}_2,\ c_2 = 0; \quad \underline{k}_1 - \underline{k}_2 = \tfrac{1}{2}, 1, \tfrac{3}{2}, ..., \\
&C_3)\, 0 < c_1 < 1,\ c_2 = 0; \quad k_1 - k_2 = 0, \pm 2, \pm 4, ..., \\
&C_4)\, 0 < c_1 < \tfrac{1}{2},\ c_2 = 0; \quad k_1 - k_2 \equiv \tfrac{1}{2} (\operatorname{mod} 2) \ \text{ or } \ \tfrac{3}{2} (\operatorname{mod} 2)\,.
\end{aligned}
\tag{37}
$$

These are chosen such that the representations are unitary and that there exists a positive scalar product. A series of $\overline{SL}(4, \mathbb{R})$ is fixed by any combination of (A), (B_i) and (C_j) $(i, j = 1, 2, 3, 4)$. For each series one can determine the k_1-k_2-sublattices. In principle, there are eight lattices

$$
\begin{aligned}
&L_1 = L(0,0), L_2 = L(\tfrac{1}{2}, \tfrac{1}{2}), L_3 = L(0,1) = L(1,0), \\
&L_4 = L(\tfrac{1}{2}, \tfrac{3}{2}) = L(\tfrac{3}{2}, \tfrac{1}{2}), L_5 = L(\tfrac{1}{2}, 0), L_6 = L(0, \tfrac{1}{2}), \\
&L_7 = L(0, \tfrac{3}{2}), L_8 = L(\tfrac{3}{2}, 0)\,.
\end{aligned}
\tag{38}
$$

In Fig. 8 only the minimal values $(\underline{k}_1, \underline{k}_2)$ of these lattices are plotted. All other points of the k_1-k_2-lattices can be obtained by performing '$\times$'-like transitions starting from the minimal values $(\underline{k}_1, \underline{k}_2)$. For the combination AB_1C_1, e.g., we have neither restrictions on k_1 nor on k_2. Thus all eight lattices are allowed, see the first diagram in the upper left corner of Fig. 8. While for the series AB_1C_i and AB_iC_1 $(i = 2, 3, 4)$ there is just one constraint, for the remaining series k_1 and k_2 have to satisfy two constraints.

Knowing the allowed k_1-k_2-lattices, we can determine the (j_1, j_2)-content. Each point (k_1, k_2) denotes all allowed (j_1, j_2), i.e. $j_1 \geq |k_1|$ and $j_2 \geq |k_2|$.

Altogether we find nine series, cf. Fig. 8, which admit the k_1-k_2-lattices L_5, L_6, L_7, and L_8. These lattices lead to the relevant j_1-j_2-lattices $L_1(1/2, 0)$ and $L_2(0, 1/2)$ of Fig. 5. For example, we could choose the so called *principal series* - the combination AB_1C_1:

$$
\pi = D^{\text{prin}}_{\overline{SL}(4,R)}(c_2, d_2, e_2; (\tfrac{1}{2}, 0)) \oplus D^{\text{prin}}_{\overline{SL}(4,R)}(c_2, d_2, e_2; (0, \tfrac{1}{2}))\,. \tag{39}
$$

However, each series, corresponding to one of the combination AB_iC_j $(i, j \neq 3)$ (9 possibilities), can be taken for the embedding.

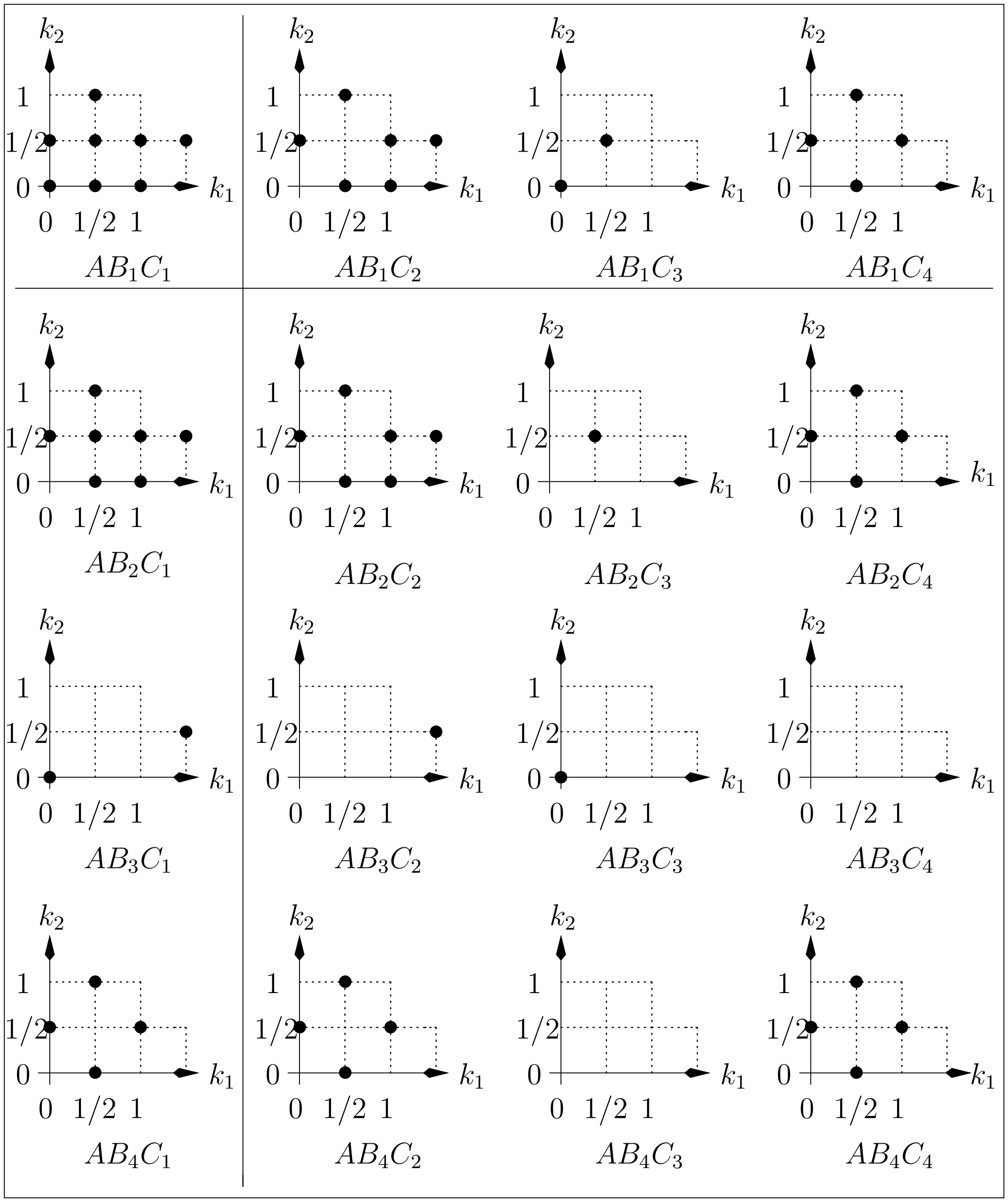


Figure 8: The k_1-k_2-lattice can be divided into eight sublattices. The 16 diagrams show the possible sublattices for each series.

9 Dirac-type field equations, minimal coupling of gravity and symmetry breaking

In this final section we want to review the steps toward affine generalization of the Dirac equation as well as its coupling to gravity. Furthermore, we propose a spontaneous symmetry breaking scenario of the $\overline{SA}(4,\mathbb{R})$ gauge symmetry down to the Poincaré one.

We started in flat 4-dimensional Minkowski spacetime. In Sections 3 to 5 we showed how Gel'fand's method to derive gamma matrices can be generalized to obtain Dirac-type equations for fermions with arbitrary spin j,

$$(i\eta^{\alpha\beta}X^{(j)}_{\alpha}\partial_{\beta} - m^{(j)})\,\Psi^{(j)} = 0\,. \tag{40}$$

The matrix $X^{(j)}_{\alpha}$ can be constructed by applying Gel'fand's method to the representation ρ_j given by Eq. (25).

In Section 6 we summed up these representations over all half-integral spin values, cf. Eq. (30), in order to describe systems such as two exchange-degenerate Regge trajectories. Spin excitations of the Regge resonances can then be introduced by minimal coupling of the Christoffel-type connection of Chromogravity. This connection is in its turn given in terms of the chrometric field $G_{\alpha\beta}$, i.e. in the anholonomic notation it reads

$$\Gamma^{\{\}\alpha}_{\beta\gamma} = \frac{1}{2}G^{\alpha\delta}(\partial_{\gamma}G_{\beta\delta} + \partial_{\beta}G_{\gamma\delta} - \partial_{\delta}G_{\beta\gamma}). \tag{41}$$

The corresponding curved space Dirac-type equation is given by

$$(iX^{\alpha}e^{i}{}_{\alpha}D_{i} - \kappa)\,\Psi = 0\,, \tag{42}$$

with the holonomic covariant derivative defined by

$$D_i = \partial_i + \Gamma^{\{\}}_{i\alpha\beta}L^{\alpha\beta}\,. \tag{43}$$

This equation is invariant with respect to local Poincaré transformations.

Since the Regge resonances can be classified by the group $\overline{SL}(4,\mathbb{R})$, in Section 8 we embedded the Regge representation ρ into a suitable representation of $\overline{SL}(4,\mathbb{R})$. Note that the spin content of a genuine world spinor field is described by the $\overline{SL}(4,\mathbb{R})$ representations as well. Formally, we are now allowed to replace the Lorentz spinor Ψ in (42) by the manifield $\overset{\text{(SL)}}{\Psi}$ spanning

the representation space of a representation of the series π defined in (39). Thus, we obtain a manifield description that is suitable for either an effective baryonic field of Regge recurrences or for a world spinor field of affine gravity.

As argued in Section 2, in a completely affine wave equation the mass term vanishes, i.e. the equation has to be of the form

$$i\widetilde{X}^{\alpha} e^{i}{}_{\alpha} D_i \overset{(\mathrm{SL})}{\Psi} = 0 \tag{44}$$

with the $\overline{SL}(4,\mathbb{R})$ vector operator $\widetilde{X}_{\alpha}$ defined by (4). In the gravity case, the covariant derivative D_i now contains a full affine connection which we take from metric-affine gravity (MAG) [5].

Note that in an equation of the form of Eq. (42) we have not specified the mass term κ so far. In order to gain (42) from (44), we propose, along the lines of Ref. [13] a symmetry breaking scenario of the $\overline{SA}(4,\mathbb{R})$ which preserves the Poincaré symmetry. It is the minimal spontaneous symmetry breaking scheme in which, besides the infinite-component $\overset{(\mathrm{SL})}{\Psi}(x)$ field, we introduce an additional 10-component second-rank symmetric $\overline{SL}(4,\mathbb{R})$ field $\varphi_{\alpha\beta}(x)$. The $\varphi_{\alpha\beta}$ field is the minimal field that (i) has non-trivial $\overline{SL}(4,\mathbb{R})$ transformation properties and (ii) it contains a Lorentz scalar component, $\varphi^{(0,0)}(x) = \eta^{\alpha\beta}\varphi_{\alpha\beta}(x)$, thus preserving the Lorentz symmetry in the process of spontaneous breaking of the $\overline{SL}(4,\mathbb{R})$ symmetry. The Lorentz decomposition of the $\varphi_{\alpha\beta}(x)$ field is $\varphi_{\alpha\beta}(x) = \varphi^{(0,0)}_{\alpha\beta}(x) + \varphi^{(1,1)}_{\alpha\beta}(x)$, where $\varphi^{(1,1)}_{\alpha\beta}(x)$ is the traceless 9-component field.

We consider the Lagrangian

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\mathrm{MAG}} + \overset{(\mathrm{SL})}{\overline{\Psi}} i\widetilde{X}^{\alpha} e^{i}{}_{\alpha} D_i \overset{(\mathrm{SL})}{\Psi} + \frac{1}{2}\eta^{\alpha\beta} e^{i}{}_{\alpha} e^{j}{}_{\beta} (D_i\varphi^{\gamma\delta})(D_j\varphi_{\gamma\delta}) \\ - \mu_{\mathrm{M}} \overset{(\mathrm{SL})}{\overline{\Psi}} \varphi^{\gamma\delta}\varphi_{\gamma\delta} \overset{(\mathrm{SL})}{\Psi} - \frac{\mu^2}{2}\varphi^{\gamma\delta}\varphi_{\gamma\delta} - \frac{\lambda}{4}(\varphi^{\gamma\delta}\varphi_{\gamma\delta})^2 \,, \end{aligned} \tag{45}$$

which describes manifield $\overset{(\mathrm{SL})}{\Psi}$, 10-component field $\varphi_{\alpha\beta}$, their mutual interaction, as well as their affine gravity interactions. Here $\varphi_{\alpha\beta}$ interacts with the manifield with strength μ_{M} and $\mathcal{L}_{\mathrm{MAG}}$ is the most general MAG Lagrangian given by Eq. (10) in [6]. Provided $\mu^2 < 0$, one finds a non-trivial vacuum expectation value determined by

$$\lambda\langle 0|\varphi^{\gamma\delta}\varphi_{\gamma\delta}|0\rangle + \mu^2 = 0. \tag{46}$$

We perform a suitable $\overline{SL}(4,\mathbb{R})$ transformation in the space of field components, such that $\varphi^{\gamma\delta}\varphi_{\gamma\delta} = \varphi^{(0,0)\gamma\delta}\varphi^{(0,0)}_{\gamma\delta}$, and obtain the nontrivial vacuum expectation value for the Lorentz scalar component, $v \equiv \langle 0|\varphi^{(0,0)}|0\rangle = \sqrt{-\mu^2/\lambda}$.

Taking $\varphi_{\alpha\beta}(x) = (v+\chi^{(0,0)}(x))\eta_{\alpha\beta}+\varphi^{(1,1)}_{\alpha\beta}(x)$, we find a massive scalar field $\chi^{(0,0)}$, and a set of nine massless Goldstone fields $\varphi^{(1,1)}_{a\beta}$, while the spinorial manifield acquires mass as well, i.e.

$$m(\chi^{(0,0)}) = \sqrt{-2\mu^2}\,, \quad m(\varphi^{(1,1)}) = 0\,, \quad m(\overset{(\mathrm{SL})}{\Psi}) = \mu_{\mathrm{M}} v^2 =: \kappa. \tag{47}$$

Let us parametrize now $\varphi_{\alpha\beta}$ as follows,

$$\varphi_{\alpha\beta}(x) = (v+\chi^{(0,0)}(x))\eta_{\mu\nu}\, \exp(\tfrac{i}{v}\chi^{(1,1)}_{\gamma\delta}T^{\gamma\delta})^{\mu}{}_{\alpha}\exp(\tfrac{i}{v}\chi^{(1,1)}_{\gamma\delta}T^{\gamma\delta})^{\nu}{}_{\beta}, \tag{48}$$

where $T^{\gamma\delta}$ are the shear generators. After the gauge transformation $U = \exp(-\frac{i}{v}\chi^{(1,1)}_{\gamma\delta}T^{\gamma\delta})$, the connection fields become (infinitesimally)

$$\Gamma'_{i(\alpha\beta)} = \Gamma_{i(\alpha\beta)} - \frac{1}{v}\partial_i\chi^{(1,1)}_{\alpha\beta}\,, \tag{49}$$

while the nine Goldstone fields $\chi^{(1,1)}_{\alpha\beta}$ get absorbed by the symmetric part of the connection $\Gamma_{i(\alpha\beta)}$ which is associated with nonmetricity. The latter in turn becomes massive, i.e. $M(\Gamma_{i(\alpha\beta)}) \neq 0$. The antisymmetric part of the connection, which is associated with spin, remains massless, i.e. $M(\Gamma_{i[\alpha\beta]}) = 0$.

We can, furthermore, make use of the nonlinear symmetry realizations and find explicitly matrix elements of the Lorentz vector X^{α}_{AB} in terms of matrix elements of the $\overline{SL}(4,\mathbb{R})$ vector $\widetilde{X}^{\alpha}_{\tilde{A}\tilde{B}}$, i.e.,

$$\begin{aligned} &X^{\alpha}_{AB} \equiv E^{\tilde{C}}{}_{A}\widetilde{X}^{\alpha}_{\tilde{C}\tilde{D}}E^{\tilde{D}}{}_{B}, \quad E^{\tilde{A}}{}_{B} = \exp(\tfrac{i}{2}\chi^{(1,1)}_{\alpha\beta}T^{\alpha\beta})^{\tilde{A}}{}_{B}\,, \\ &\Psi_A = E^{\tilde{A}}{}_{A}\overset{(\mathrm{SL})}{\Psi}{}_{\tilde{A}}, \end{aligned} \tag{50}$$

where $E^{\tilde{A}}{}_{B}$, is the nonlinear symmetry realizer. The (tracefree part of the) MAG-metric tensor $g_{\alpha\beta}$ can be defined from the Goldstone fields $\chi^{(1,1)}_{\alpha\beta}$ as

$$g_{\alpha\beta} := r^{\mu}{}_{\alpha}r^{\nu}{}_{\beta}\eta_{\mu\nu}, \quad r^{\mu}{}_{\alpha} := \exp(\tfrac{i}{2}\chi^{(1,1)}_{\alpha\beta}T^{\alpha\beta})^{\mu}{}_{\alpha} \tag{51}$$

as suggested by the nonlinear realization of the local affine group [21].

To summarize, we break spontaneously the $\overline{SL}(4,\mathbb{R})$ symmetry down to the Lorentz symmetry, the fermionic fields acquire nontrivial mass, and all quantities of an equation of the form given by Eq. (42) are explicitly given in terms of the quantities of Eq. (44).

Acknowledgments

One of the authors (I.K.) is grateful to F. W. Hehl (University of Cologne) for many discussions related to this work.

Appendix

Transition from spherical to Cartesian tensors

It is often useful to relate the Cartesian generators $L_{\alpha\beta} = M_{\alpha\beta} + T_{\alpha\beta}$ of $\overline{SL}(4,\mathbb{R})$ to the spherical tensors $J^{(1)}_\alpha$, $J^{(2)}_\alpha$ and the double tensor $Z_{\alpha\beta}$ $(\alpha,\beta = 0,\pm1)$. The inverse of Eq. (2.3) in [17] yields the generators of the maximal compact subgroup $\overline{SO}(4)$,

$$M_{ab} = \varepsilon_{abc}(J^{(1)}_c + J^{(2)}_c), \tag{52}$$

$$T_{0a} = J^{(1)}_a - J^{(2)}_a\,. \tag{53}$$

The relation between the spherical vector $J_{0,\pm}$ and the Cartesian vector J_a are well-known.

We decompose the double tensor $Z_{\alpha\beta}$ of rank $(1,1)$ with respect to the rotation group, $\overline{SO}(4) \supset \overline{SO}(3)$, $D^{(1)} \times D^{(1)} = D^{(0)} \oplus D^{(1)} \oplus D^{(2)}$, and obtain the three corresponding tensors

$$Z^{(k)}_\gamma = \sum_{\alpha,\beta} Z_{\alpha\beta}\,(11\alpha\beta|11k\gamma)\,, \tag{54}$$

cf. Eq. (35.2) in [22], with rank $k = 0,1,2$ $(\gamma = -k,...,+k)$ and the Clebsch-Gordon coefficient $(11\alpha\beta|11k\gamma)$. The tensors $Z^{(0)}_\gamma$, $Z^{(1)}_\gamma$, and $Z^{(2)}_\gamma$ have 1, 3, and 5 independent components which we now relate to the Cartesian tensor Z_{ab},

$$Z_{31} = -\frac{1}{2}(Z^{(2)}_{+1} - Z^{(2)}_{-1} + Z^{(1)}_{+1} + Z^{(1)}_{-1})\,,$$

$$Z_{13} = -\frac{1}{2}(Z^{(2)}_{+1} - Z^{(2)}_{-1} - Z^{(1)}_{+1} - Z^{(1)}_{-1})\,,$$

$$Z_{23} = \frac{i}{2}(Z^{(2)}_{+1} + Z^{(2)}_{-1} + Z^{(1)}_{+1} - Z^{(1)}_{-1})\,,$$

$$Z_{32} = \frac{i}{2}(Z^{(2)}_{+1} + Z^{(2)}_{-1} - Z^{(1)}_{+1} + Z^{(1)}_{-1})\,,$$

$$
\begin{aligned}
Z_{12} &= -\frac{i}{2}(Z^{(2)}_{+2} - Z^{(2)}_{-2} + \sqrt{2} Z^{(1)}_0)\,, \\
Z_{21} &= -\frac{i}{2}(Z^{(2)}_{+2} - Z^{(2)}_{-2} - \sqrt{2} Z^{(1)}_0)\,, \\
Z_{11} &= \frac{1}{2}(Z^{(2)}_{+2} + Z^{(2)}_{-2}) - \frac{1}{\sqrt{6}} Z^{(2)}_0 - \frac{1}{\sqrt{3}} Z^{(0)}_0\,, \\
Z_{22} &= -\frac{1}{2}(Z^{(2)}_{+2} + Z^{(2)}_{-2}) - \frac{1}{\sqrt{6}} Z^{(2)}_0 - \frac{1}{\sqrt{3}} Z^{(0)}_0\,, \\
Z_{33} &= \frac{2}{\sqrt{6}} Z^{(2)}_0 - \frac{1}{\sqrt{3}} Z^{(0)}_0\ .
\end{aligned} \tag{55}
$$

Z_{ab} is related to the spatial shear tensor T_{ab} and to the boosts M_{0c} according to

$$
Z_{ab} = T_{ab} + \varepsilon_{abc} M_{0c}\,. \tag{56}
$$

Non-minimal solution for the multiplicities

For the multiplicities $M_i = n+2-i$ of the representations τ_i ($i = 1, ..., n+1$), we have to show that $A_l \geq B_l$ for all $l = \frac{1}{2}, ..., j$.

Proof by induction: Since $A_j = B_j = 1$, $A_l \geq B_l$ for $l = j$. Now, assume $A_l \geq B_l$. Using Rules 1 to 4, we obtain

$$
\begin{aligned}
A_{l-1} &= A_l + M_i(M_i+1) + \frac{(M_i+1)(M_i+2)}{2} - B_l \\
&\geq (M_i+1)(\frac{3}{2}M_i + 1) \qquad (57) \\
B_{l-1} &= B_l + M_i + 1 \qquad (58)
\end{aligned}
$$

with M_i being the multiplicity of $\tau_i, i = l + 1/2$, and $M_{i-1} = M_i + 1$ the multiplicity of τ_{i-1}.

Now $A_{l-1} \geq B_{l-1}$ follows since

$$
(M_i+1)\frac{3}{2}M_i \geq B_l = \sum_{k=i}^{n+1} M_k = \frac{1}{2} M_i (M_i+1)\,. \tag{59}
$$

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